Recursive Generation and Parametric Modelling of Two Dimensional Sinewaves

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Abstract

This communication deals with the analysis of an image representing a finite sum of two dimensional sinewaves or complex exponentials in presence of a stationary white noise. The purpose is the estimation of the characteristics of these sinewaves (bidimensional frequency and complex amplitude). This estimation is based on the Prony Pisarenko method and requires a bidimensional extension of this approach. Then the estimated sinewaves are generated in using a recursive (autoregressive) model.

1. Introduction

In the one-dimensional case, the Prony Pisarenko method ([1], [2], [4]) is based on a parametric linear recursive representation of a signal in order to model it as a sum of real or exponentials (in the case of undamped sinewaves, the exponential are imaginary) to which is added a stationary zero mean white noise. The estimation of the parameters consists in the computation of the signal covariance matrix, followed by the computation of the eigenvector associated with its smallest eigenvalue. When these recursive model (filter) parameters are obtained (the frequential characteristics), the complex amplitudes are computed. This second step begins with the computation of the filter impulse response. Then the estimation of the initial conditions of the generation of the sinewaves (this information is equivalent to the complex amplitude of the components) consists in finding the coefficients of the delayed impulse responses through the minimization of the squared error between the analyzed signal and this sum of delayed and weighted impulse responses.

In order to extend this approach to the case of two dimensional signals, it is necessary to specify the support of the recursive filter allowing the generation of the signal from initial conditions. The main point of the communication is the proposition of a double form of the support allowing the analysis/synthesis of the signal. It is related to the recursive vectorial representation of images [3]. This specific representation of bidimensional signals is described in the second section. The third section details the bidimensional extension of the Prony Pisarenko estimatition method yielding the recursive model. The fourth section illustrates the proposition through simulations.

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2. Recursive representation of bidimensional sinewaves

Let $f(x, y)$ be the sum of p bidimensional (or complex exponential) sinewaves

$$f(x,y) = \sum_i g(i) \exp(u_i x + v_i y) \cdot \tag{1}$$

In this sum, the index i varies from 1 to p, and the coefficients $g(i)$, u_i and v_i are complex; u_i and v_i are pure imaginary in the case of undamped sinewaves. How to generate recursively $f(x,y)$ from initial conditions? This question has several answers and it is necessary to choose a particular form of the recursive representation of the signal and to define the domain where are given the initial conditions from which the signal will be generated. In the proposed approach, the first step in the generation of the signal $f(x,y)$ will be the generation of one line from initial conditions using a one-dimensional recursion ; the second step will use a second (bidimensional) recursion in order to generate the two lines above and below the first one, and next the other lines.

2.1 Recursive representation of the lines in the image

On one line of the image (y constant equal to y_0), the expression (1) is monodimensional

$$f(x, y_0) = \sum_i g(i) \exp(u_i x + v_i y_0) = \sum_i g(i) \exp(v_i y_0) \exp(u_i x) \tag{2}$$

In this special case, the only visible frequencies are the ‘horizontal’ frequencies u_i; the information concerning the ‘vertical’ frequencies v_i appears as a modification of the initial conditions that depend on the line under consideration. The initial conditions are given on p successive points of one line, that is for the values 0,..., p-1 of x and for the null value of y. The recursive generation of that line of the image from these initial conditions requires the computation of the recursive filter with poles $\exp(u_i)$, that is

$$f(x, y_0) = -\sum_i a(i) f(x-i, y_0) \tag{3}$$

This recursive filter (that is neither stable nor causal in the general case) allows the generation of the values of $f(x, y_0)$ for increasing values of x starting from p. It can also be used in order to generate the signal for decreasing negative values of x, assuming that the model order is correct, ($a(p)$ is different from zero and $a(0)$ equals one).

$$f(x, y_0) = -\frac{1}{a(p)} \sum_i a(p-i) f(x+i, y_0) = -\sum_i a'(p-i) f(x+i, y_0), \tag{4}$$

This procedure generates a whole line from the p initial values.

2.2 Recursive generation of the other lines of the image

In order to generate the other lines of the image, starting from this line with null ordinate, it is necessary to choose a form of the recursion allowing this computation. A simple proposition is the following: the generation of the line of ordinate $y = 1$ and subsequently of the lines of positive ordinate will use a computation of the form

$$f(x, y) = -\sum_i b(i) f(x - i, y - 1) \quad (5)$$

where the recursion order is equal to p and i varies from 1 to p. The computation of $f(x, y)$ for the negatives ordinates requires a modification of this formula that will be detailed later. This bidimensional computation may be performed only on p points of the new line, the recursive horizontal computation allowing the generation of the other samples of the line.

In order to ensure that the p coefficients $b(i)$ generate recursively the sum of the p sinewaves $\sum_i g(i) \exp(u_i x + v_i y)$, these coefficients must satisfy the following set of p linear equations

$$\begin{gathered}
\exp v_1 + \exp - u_1 . b(1) + \ldots + \exp - p u_1 . b(p) = 0 \\
\exp v_2 + \exp - u_2 . b(1) + \ldots + \exp - p u_2 . b(p) = 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\exp v_p + \exp - u_p . b(1) + \ldots + \exp - p u_p . b(p) = 0
\end{gathered} \quad (6)$$

The generation of the points in negative ordinate lines uses the formula

$$f(x, y) = -\sum_i b'(i) f(x - i, y + 1) \quad (7)$$

where the predictor/generator coefficients $b'(i)$ must satisfy

$$\begin{gathered}
\exp - v_1 + \exp - u_1 . b'(1) + \ldots + \exp - p u_1 . b'(p) = 0 \\
\exp - v_2 + \exp - u_2 . b'(1) + \ldots + \exp - p u_2 . b'(p) = 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\exp - v_p + \exp - u_p . b'(1) + \ldots + \exp - p u_p . b'(p) = 0
\end{gathered} \quad (8)$$

It is also possible to predict or compute the sample $f(x, y)$ from the samples $f(x+i, y-1)$ (predictor b'') or the samples $f(x+i, y+1)$ (predictor b'''), yielding similar formulas for the computation of a sample at the left of the upper line. The coefficients of the predictor $b''(i)$ must satisfy

$$\begin{aligned} &\exp v_1 + \exp u_1 . b''(1) + \ldots + \exp pu_1 . b''(p) = 0 \\ &\exp v_2 + \exp u_2 . b''(1) + \ldots + \exp pu_2 . b''(p) = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\exp v_p + \exp u_p . b''(1) + \ldots + \exp pu_p . b''(p) = 0 \end{aligned} \tag{9}$$

The prediction/generation of a sample at the left of the line is given by the second expression of formula (4) where the predictor coefficients $a'(i)$ satisfy

$$\begin{aligned} &1 + \exp u_1 . a'(1) + \ldots + \exp pu_1 . a'(p) = 0 \\ &1 + \exp u_2 . a'(1) + \ldots + \exp pu_2 . a'(p) = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &1 + \exp u_p . a'(1) + \ldots + \exp pu_p . a'(p) = 0 \end{aligned} \tag{10}$$

For the generation of a sample on the left of the lower line, the predictor coefficients $b'''(i)$ must satisfy

$$\begin{aligned} &\exp v_1 + \exp - u_1 . b'''(1) + \ldots + \exp - pu_1 . b'''(p) = 0 \\ &\exp v_2 + \exp - u_2 . b'''(1) + \ldots + \exp - pu_2 . b'''(p) = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\exp v_p + \exp - u_p . b'''(1) + \ldots + \exp - pu_p . b'''(p) = 0 \end{aligned} \tag{11}$$

When the horizontal frequencies u_i and the coefficients $b(i)$ of the predictor are known, equations (6) gives the 'vertical' frequencies v_i. So, three of the four bidimensional predictors (*b*), (*b'*), (*b''*) or (*b'''*) can be deduced from the fourth and one of the two horizontal predictors through the solution of a set of linear equations (see figure 1)

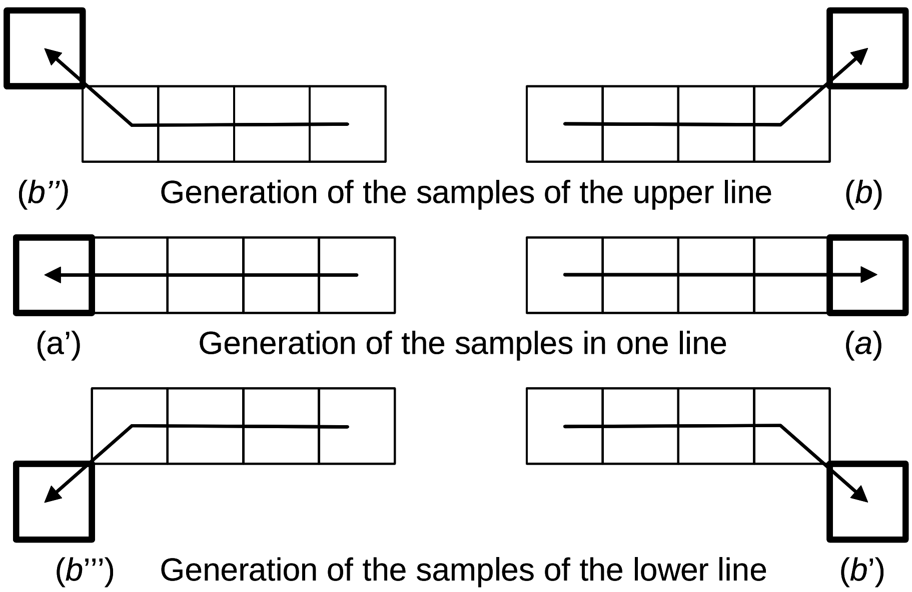

Figure 1. Prediction/generation pattern in the image

Remarks on possible extensions: The approach extends directly to a three-dimensional space in considering the prediction of one line in the third dimension just as in the second dimension; then there are 10 predictors instead of 6. This three dimensional extension may be useful for the analysis of wave propagation. It is also possible to extend the approach to the case where the initial conditions are given on a rectangular support and not a linear segment; then the number of predictors is equal to à q + 2, where q is the height of the rectangle.

2.3 Limitations in terms of accuracy

If the propagation direction of the sinewaves is close to the vertical, the recursive representation where the initial conditions are given on a horizontal line may yield inaccurate results. Then the initial conditions and the successive samples must be given on a vertical line. In some cases, it will be necessary to separate the frequency domain in two regions and process separately the components close to the horizontal direction on one side, and the components close to the vertical direction on the other.

3. Prony Pisarenko method in two dimensions

Following the recursive generation procedure, this analysis will be performed in two steps: firstly, the horizontal analysis of the lines and secondly, the bidimensional extension.

3.1 The one-dimensional analysis

The estimation of the horizontal frequencies u_i reduces to the monodimensional problem. The generation of one line in the image (y = 0) requires the knowledge of p initial conditions in the positions i = 0,..., p-1. These two steps are solved in using the Prony Pisarenko method: the computation of the coefficients a_i of the predictor begins with the computation of the covariance matrix with term $r_{k,l}$

$$r_{k,l} = \sum_{x,y} f(x-k, y)\overline{f(x-l, y)} , \tag{12}$$

then the eigenvector associated with the lowest eigenvalue of this matrix is computed yielding the horizontal predictor. The second step, corresponding to the computation of the initial conditions will be described later. The computation of a covariance matrix of a signal that is not stationary yields a recursive model generating exponentials and damped sinewaves as well as pure sinewaves, which are obtained in the stationary case, that is when the covariance matrix is Toeplitz, which is the case considered by Pisarenko.

3.2 Adjunction of the second dimension

The present recursive formulations allows the application of the Prony Pisarenko method after a modification of the construction of the covariance matrix: the measured signal $s(x, y)$ is the sum of the signal $f(x, y)$ satisfying the recursive equation and of a white noise $w(x, y)$ independent of $f(x, y)$

$$s(x, y) = f(x, y) + w(x, y) \tag{13}$$

Using this expression in order to replace *f*(*x*, *y*) in equation (5) yields

$$s(x,y) - w(x,y) + \sum_i b(i)\big(s(x-i,y-1) - w(x-i,y-1)\big) = 0 \tag{14}$$

The multiplication of this equation by the complex conjugates of *s*(*x*, *y*) and *s*(*x*-1, *y*-1), ..., *s*(*x*-*p*, *y*-1) followed by the computations of means show that the measured signal must satisfy

$$\begin{pmatrix} E\left(s(x,y)\overline{s(x,y)}\right) & E\left(s(x-1,y-1)\overline{s(x,y)}\right) & \cdots & E\left(s(x-p,y-1)\overline{s(x,y)}\right) \\ E\left(s(x,y)\overline{s(x-1,y-1)}\right) & E\left(s(x-1,y-1)\overline{s(x-1,y-1)}\right) & \cdots & E\left(s(x-p,y-1)\overline{s(x-1,y-1)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ E\left(s(x,y)\overline{s(x-p,y-1)}\right) & E\left(s(x-1,y-1)\overline{s(x-p,y-1)}\right) & \cdots & E\left(s(x-p,y-1)\overline{s(x-p,y-1)}\right) \end{pmatrix} \times \begin{pmatrix} 1 \\ b(1) \\ \vdots \\ b(p) \end{pmatrix} = \begin{pmatrix} 1 \\ b(1) \\ \vdots \\ b(p) \end{pmatrix} \sigma^2 \tag{15}$$

where the real eigenvalue σ^2 is the variance of the stationary measurement noise *w*(*x*, *y*); the subtraction of the noise is possible only if this eigenvalue is the smallest of the eigenvalues of the covariance matrix. This equation gives the bidimensional predictor which, associated with the line monodimensional predictor, allows the correct representation of the analyzed image. As mentioned at the end of section 2.2, this knowledge of both predictors allows the recursive generation of the image starting from initial conditions; it also gives the full spatial frequencies with both the horizontal and vertical components;

3.3 Estimation of the initial conditions

Although the proposed criterion is not consistent with the hypotheses on which the recursive model is based, it may be useful to complete the approach with a step where the signal is synthesized in order, for instance to smooth the signal.

3.3.1 Computation of the modes

As the bidimensional frequencies are known, the analyzed signal can be matched with a sum of weighted exponentials. These exponentials have the value one at the origin of the plane and they can be generated recursively using the following formula and its three equivalent expressions in the negative horizontal direction and the positive and negative vertical directions

$$\exp\big(u_i(x+1) + v_i y\big) = \exp\big(u_i x + v_i y\big).\exp\big(u_i\big) \tag{16}$$

3.3.2 Formalization and minimization of the quadratic criterion

s(*x*, *y*) will be constructed as the sum of *p* bidimensional exponentials to which is added a noise $w'(x,y)$

$$s(x,y) = \sum_i c(i)\exp(u_i x + v_i y) + w'(x,y) \tag{17}$$

Then the computation of the coefficients *c*(*i*) can be reduced to the minimization of a quadratic criterion (the variance of $w'(x,y)$), although this criterion is not necessarily compatible with the measurement noise whiteness assumption on which the first step of the identification process is based. When the *c*(*i*) have been computed, the initial values are obtained in applying (16) with a null noise. However this second step is very dependant on the accuracy of the recursive models parameters: when the noise level is high, the covariance matrix estimation is not perfect and the evaluation of the characteristics of the exponential components is approximate; this lack of accuracy (especially on its real part associated with the damping factor) can severely deteriorate the final results. So, it may require improvements as proposed below.

3.4 Possible improvement

In order to reduce the inaccuracies occurring in this estimation procedure, a gradient technique can be used (with care, since the formulas are highly nonlinear). This may be done in using the following recursive procedure:

At step t of the recursion, there is an estimation of the parameters ($c(i)$, u_i, v_i) on a rectangular portion of the image (with size XxY) ; the size of this portion of the image is slightly increased to (X+ΔX)x(Y+ ΔY) ; the gradient of the sum of the squares of the differences between $f(x,y)$ and its prediction $s(x,y)$ is computed and the linear corrections $\Delta c(i)$, Δu_i, Δv_i are computed in order to minimize the quadratic form

$$\sum_{x=0}^{X}\sum_{y=0}^{Y}\left| f(x,y - \sum_i \exp(u_i x + v_i y)\big(c(i)(1 + x\Delta u_i + y\Delta u_i) + \Delta c(i)\big)\right|^2$$

This linear correction may be repeated several times; then the domain XxY is slightly extended again, and the correction of the parameters is repeated. However, an excessive augmentation of the domain size may yield a local minimum.

4. Confirmation by simulations

The analysis was performed on a 256x256 image composed of two complex undamped sinewaves. Their horizontal and vertical frequencies were u_0=80.2/256 and v_0=-60.3/256 (or 195.7/256) for the first and 30.4/256 and 70.5/256 for the second, the sampling frequency being equal to 1; their amplitudes were g_0=0.7 and g_1=1. The results are visualized in the frequency domain in fig. 2. The effects of the noise level are summarized in table I. Figure 3 presents the results of the improvement, when the analyzed signal is in the very low frequency range; then the Prony Pisarenko method yields rather poor results that are improved by a few steps of the error minimization proposed in paragraph 3.4. It should be stressed that this step has to be applied with care, since the control of its convergence is quite difficult.

5. Conclusion

Although there exist numerous sophisticated methods (see for instance [5], [6], [7], [8] among others), the proposed one may present some interest at least as an academic exercise: its implementation is rather simple. It also presents limitations: when the noise level is high, the accuracy of the covariance matrix estimation is not perfect and the evaluation of the characteristics of the exponential components is approximate; this lack of accuracy (especially on its real part associated with the damping factor) has a negative effect on the estimation of the initial conditions. It can be reduced by a refinement of the estimation of the parameters of the exponentials.

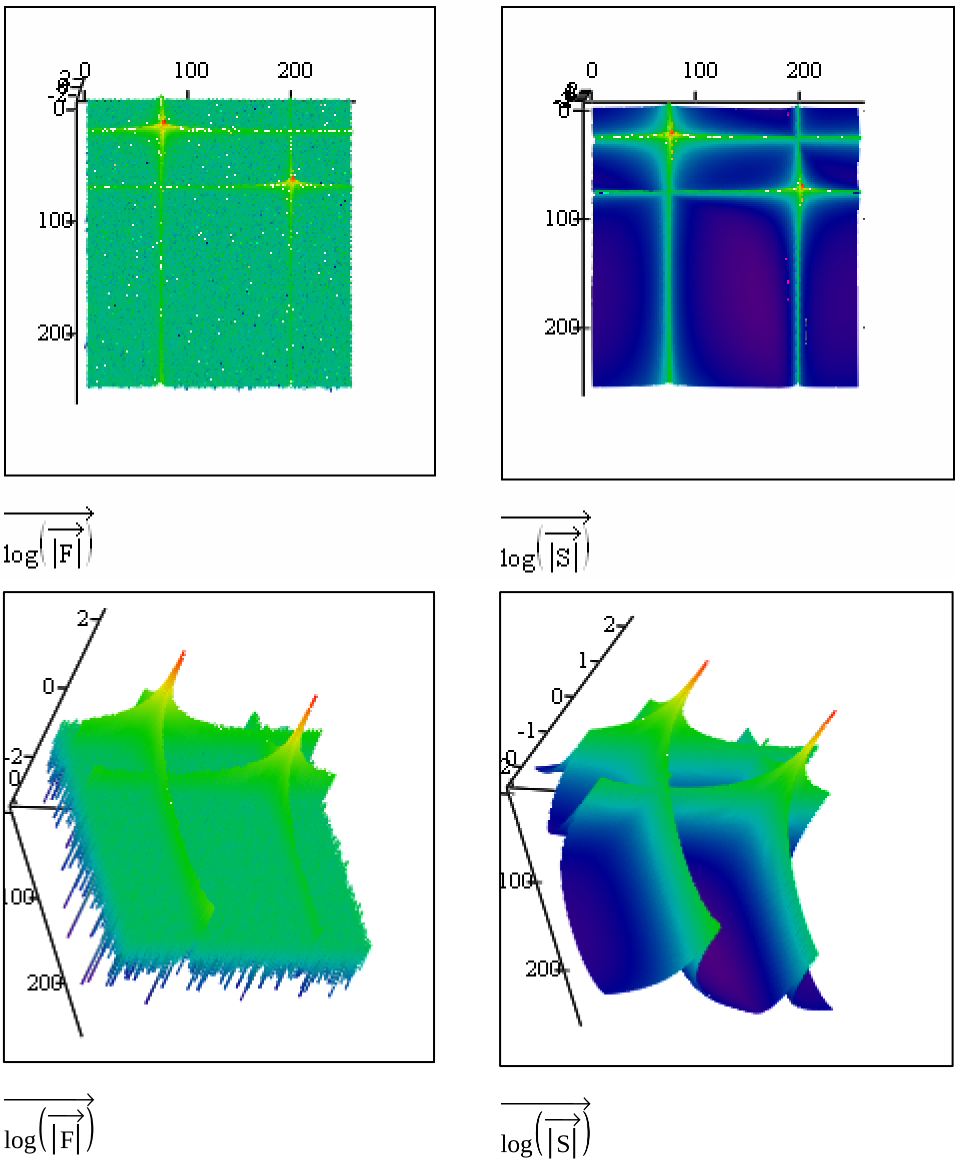

Figure 2 : Representation of the analyzed signal in the frequency domain (left) and of the reconstructed sinewaves (right) (the poor accuracy of the representation is due to the limitations of the 256x256 discrete Fourier transform)

Table I: Effect of the noise level on the parameters estimation

noise standard deviation	typical error (standard deviation) on frequency and damping	typical error (standard deviation) on initial conditions	typical error (standard deviation) on the regenerated signal
0.0001	1.38E-07	2.01E-05	1.94E-04
0.001	2.19E-06	1.84E-04	8.24E-04
0.01	2.18E-05	1.87E-03	8.34E-03
0.1	1.47E-04	0.018	0.086
1	6.81E-03	0.566	1.722
10	0.626	1.022	not significant

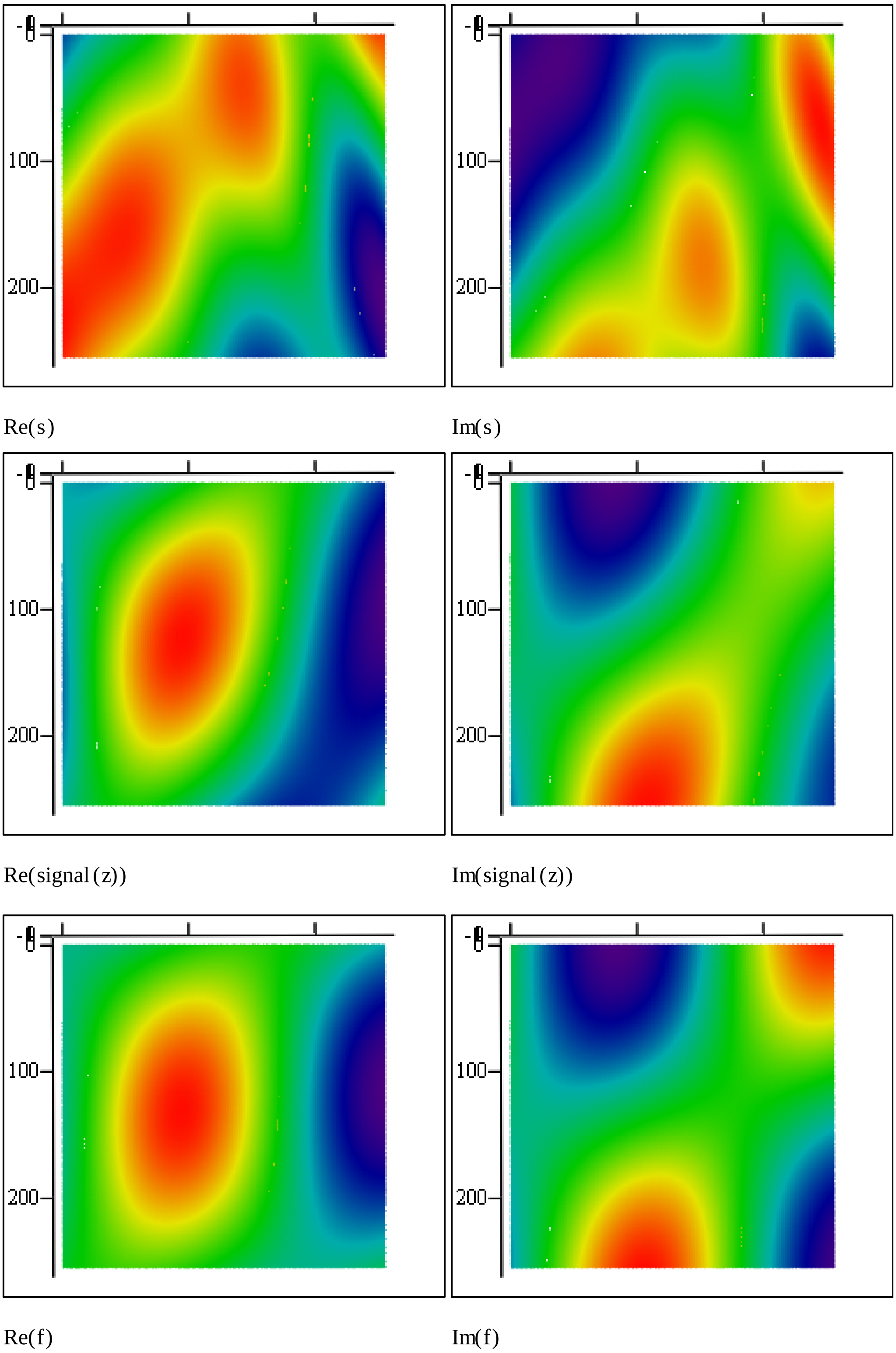

Fig. 3. Improvement by gradient correction of the parameters of the exponentials: Upper view: the signal estimation after application of Prony's method; Middle view: Evolution after 8 steps of the gradient correction (one step ahead of convergence); Lower view: the final result after the last step, coinciding with the true signal.

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